

TWISTED CYCLIC HOMOLOGY AND CROSSED PRODUCT ALGEBRAS

JACK M. SHAPIRO

ABSTRACT. $HC_*(A \rtimes G)$ is the cyclic homology of the crossed product algebra $A \rtimes G$. For any $g \in G$ we will define a homomorphism from $HC_*^g(A)$, the twisted cyclic homology of A with respect to g , to $HC_*(A \rtimes G)$. If G is the finite cyclic group generated by g and $|G| = r$ is invertible in k , then $HC_*(A \rtimes G)$ will be isomorphic to a direct sum of r copies of $HC_*^g(A)$. For the case where $|G|$ is finite and $Q \subset k$ we will generalize the Karoubi and Connes periodicity exact sequences for $HC_*^g(A)$ to Karoubi and Connes periodicity exact sequences for $HC_*(A \rtimes G)$.

For an associative unitary algebra A over a commutative ring k together with a k -algebra automorphism g , the twisted cyclic homology of A with respect to g , $HC_*^g(A)$, is defined in [2]. It comes from the homology of a bicomplex (C_*, b, B) where C_n is the quotient of the $(n+1)$ -fold tensor product of A over k , denoted by $A^{(n+1)}$, modulo the action of g . If we write $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ as (a_0, a_1, \dots, a_n) , then the action of g on $A^{(n+1)}$ is given by T_g , where

$$T_g(a_0, a_1, \dots, a_n) = (g(a_0), g(a_1), \dots, g(a_n)).$$

Given a full (discrete) group G of k -algebra automorphisms of A we can form the crossed product algebra $A \rtimes G$ and as shown in [1, Corollary 4.2], its cyclic homology $HC_*(A \rtimes G)$ can be derived from a bicomplex $(C_*, b + \bar{b}, B + T\bar{B})$ with $C_n = \sum_{p+q=n} k[G^{p+1}] \otimes A^{(q+1)}$. The action of T on $(g_0, \dots, g_p/a_0, \dots, a_q) \in k[G^{p+1}] \otimes A^{(q+1)}$ is the identity on $k[G^{p+1}]$ and T_g on $A^{(q+1)}$, where $g = g_0 \cdots g_p$. A slight alteration can be made in this bicomplex, moding out C_n by the action of T and dropping $T\bar{B}$ from the boundary maps. From this we get a homomorphism from $HC_*^g(A)$ to $HC_*(A \rtimes G)$ for any $g \in G$. For the case where G is the finite cyclic group generated by g with $|G|$ invertible in k , $HC_*(A \rtimes G)$ will be isomorphic to a direct sum of r copies of $HC_*^g(A)$, where $r = |G|$, and the homomorphism will be an isomorphism of $HC_*^g(A)$ onto one of the direct summands. It will then follow that if $\text{order}(g) = r$, with r invertible in k , then $HC_*^g(A) \simeq HC_*^{g^n}(A)$, for g raised to any power n , where $(n, r) = 1$. For the case where G is a finite group and $Q \subset k$ we can generalize the procedure used in [4] for the twisted de Rham homology of A , to define $\bar{H}DR_*^G(A)$, the G -de Rham homology of A , and from that get a Karoubi exact sequence

$$0 \rightarrow \bar{H}DR_n^G(A) \rightarrow HC_n(A \rtimes G) \rightarrow \bar{H}H_{n+1}^G(A).$$

A similar thing will be done to get a Connes periodicity exact sequence.

1. CYCLIC HOMOLOGIES ASSOCIATED TO k -ALGEBRA AUTOMORPHISMS

A first quadrant bicomplex is a collection of k -modules $C_{p,q}$ indexed by the integers $p \geq 0$ and $q \geq 0$ together with a horizontal differential $d^h : C_{p,q} \rightarrow C_{p-1,q}$ and a vertical differential $d^v : C_{p,q} \rightarrow C_{p,q-1}$ satisfying $d^h \circ d^h = d^v \circ d^v = d^h \circ d^v + d^v \circ d^h = 0$. From this we can form a chain complex, called the total complex, and its homology groups are called the homology groups of the bicomplex [3, 1.0.11]. One such bicomplex is the one used in [3, 2.1.7.1] to define cyclic homology. It comes from maps $b : C_n \rightarrow C_{n-1}$ and $B : C_n \rightarrow C_{n+1}$ satisfying $b \circ b = B \circ B = b \circ B + B \circ b = 0$. Given a k -algebra automorphism, g , of a k -algebra A , we have the Hadfield-Kr hmer bicomplex whose homology groups define the twisted cyclic homology of A with respect to g . Here $C_n = \frac{A^{(n+1)}}{(1-T_g)}$ and the maps b and B are as follows. $b = \sum_{i=0}^n (-1)^i d_i$, where for $0 \leq i \leq n-1$, $d_i(a_0, a_1, \dots, a_n) = (a_0, \dots, a_i a_{i+1}, \dots, a_n)$ and $d_n(a_0, a_1, \dots, a_n) = (g(a_n) a_0, a_1, \dots, a_{n-1})$. As in cyclic homology the formula for B is simpler in the normalized case where we get $B(a_0, a_1, \dots, a_n) = \sum_{i=0}^n (-1)^{ni} (1, g(a_i), \dots, g(a_n), a_0, \dots, a_{i-1})$ [2, section 2].

Another example of a bicomplex is the one given by Getzler-Jones in [1] to compute the cyclic homology of a crossed product algebra, $A \rtimes G$. As mentioned in the introduction $C_n = \sum_{p+q=n} k[G^{p+1}] \otimes A^{(q+1)}$, and the boundary maps are given as $b + \bar{b}$ and $B + T\bar{B}$. The maps b and B on $(g_0, \dots, g_p/a_0, \dots, a_q)$ are the extensions of the maps b and B from the Hadfield-Kr hmer complex with respect to $g^{-1} = (g_0 \cdots g_p)^{-1}$, that fix g_0, \dots, g_p in each term. The map \bar{b} is given by $\bar{b}(g_0, \dots, g_p/a_0, \dots, a_q) = \sum_{i=0}^{p-1} (-1)^i (g_0, \dots, g_i g_{i+1}, \dots, g_p/a_0, \dots, a_q) + (-1)^p (g_p g_0, \dots, g_{p-1}/g_p(a_0), \dots, g_p(a_q))$, and \bar{B} is given by $\bar{B}(g_0, \dots, g_p/a_0, \dots, a_q) = \sum_{i=0}^p (-1)^{ip} (1, g_{p-i+1}, \dots, g_p, g_0, \dots, g_{p-i}/h_i(a_0), \dots, h_i(a_q))$, $h_i \equiv g_{p-i+1} \cdots g_p$. $T(g_0, \dots, g_p/a_0, \dots, a_q) = (g_0, \dots, g_p/g(a_0), \dots, g(a_q))$, where $g = g_0 \cdots g_p$.

Given a bicomplex coming from (C_*, b, B) , together with a k -module W , the cyclic homology with coefficients in W is defined in [1, section 4] as follows. Treating W as a $k[u]$ -module with $\deg(u) = -2$ and using the degrees in C_* they form the chain complex $(C_*[[u]] \otimes_{k[u]} W, b + uB)$. The homology with coefficients in W is then defined as the homology of this chain complex. For a specific W we can write it in a simpler form, as pointed out in [1, sec. 4].

Lemma. *For $W = k[u, u^{-1}]/uk[u]$ the homology of the chain complex $(C_*[[u]] \otimes_{k[u]} W, b + uB)$ is the same as the homology of the bicomplex (C_*, b, B) .*

Proof. For this choice of W the n^{th} degree term of the chain complex, $(C_*[[u]] \otimes_{k[u]} W, b + uB)$, will be

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_{n-2j} \cdot u^{-j},$$

with boundary map $b + uB$. For the total complex of (C_*, b, B) the n^{th} degree term will be

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_{n-2j}$$

with boundary map is $b + B$. These two chain complexes are quasi-isomorphic. \square

From the above lemma it follows that for $W = k[u, u^{-1}]/uk[u]$ we can read corollary 4.2 in [1] as saying that the cyclic homology $HC_*(A \rtimes G)$ is isomorphic to the cyclic homology coming from the bicomplex

$$\left(\sum_{p+q=n} k[G^{p+1}] \otimes A^{(q+1)}, b + \bar{b}, B + T\bar{B} \right).$$

We will now show that in fact for that choice of W the result of corollary 4.2 can be restated as follows.

Proposition. *We have an isomorphism between the cyclic homology of the crossed product algebra $HC_*(A \rtimes G)$ and the homology coming from the bicomplex*

$$\left(\sum_{p+q=n} \frac{k[G^{p+1}] \otimes A^{(q+1)}}{1-T}, b + \bar{b}, B \right).$$

Proof. The bicomplex is well defined since it is shown in [1] that we have the equations $bB + Bb = 1 - T$, $\bar{b}B + B\bar{b} = 0$, and T commutes with b, \bar{b} and B . To get the E_{pq}^1 term in the spectral sequence used for the proof of [1, lemma 4.3], one of the main tools used is the isomorphism, $\beta : k[G^{p+1}] \otimes A^{(q+1)} \rightarrow k[G^p] \otimes (k[G] \otimes A^{(q+1)})$, given by $\beta : (g_0, \dots, g_p/a_0, \dots, a_q) \rightarrow (g_1, \dots, g_p/g/a_0, \dots, a_q)$, where $g = g_0 \cdots g_p$. This induces an isomorphism

$$\beta : \frac{k[G^{p+1}] \otimes A^{(q+1)}}{1-T} \rightarrow k[G^p] \otimes \frac{k[G] \otimes A^{(q+1)}}{(1-T)},$$

which is a chain map with respect to \bar{b} on the left, and the boundary for group homology with coefficients in $\frac{k[G] \otimes A^{(q+1)}}{(1-T)}$, on the right. Using this together with the filtration introduced in [1], after corollary 4.2, now applied to the total complex of the bicomplex

$$\left(\sum_{p+q=n} \frac{k[G^{p+1}] \otimes A^{(q+1)}}{1-T}, b + \bar{b}, B \right),$$

we get that $E_{pq}^1 = H_p(G, \frac{k[G] \otimes A^{(q+1)}}{1-T})$. This is quasi-isomorphic to the E_{pq}^1 term in [1] since as pointed out there, T is chain homotopic to the identity on $H_p(G, k[G] \otimes A^{(q+1)})$. The rest of the proof will then be the same as in [1]. \square

2. RELATION BETWEEN THE HOMOLOGIES

We can consider the bicomplex $(\frac{k[G] \otimes A^{(n+1)}}{1-T}, b, B)$ as a sub-bicomplex of the bicomplex of the above proposition, since \bar{b} is the zero map on $\frac{k[G] \otimes A^{(n+1)}}{1-T}$. The homology of this bicomplex will be denoted as $HC_*^G(A)$, and can be thought of as an extension of twisted cyclic homology to a full group, G .

Theorem. *If G is a group of k -algebra automorphisms of a k -algebra A then for any $g \in G$ we have a homomorphism from the twisted cyclic homology of A with respect to g into $HC_*(A \rtimes G)$, $f : HC_*^g(A) \rightarrow HC_*(A \rtimes G)$. For the case when G is the cyclic group generated by g , $|G| < \infty$ and $|G|$ is invertible in k (e.g. $ch(k)=0$), $HC_*(A \rtimes G)$ will be isomorphic to a direct sum of r copies of $HC_*^g(A)$, where $r = |G|$, and f will be an isomorphism of $HC_*^g(A)$ onto one of the summands.*

Proof. For any $g \in G$, we have a map of bicomplexes, $(\frac{A^{(n+1)}}{1-T_g}, b, B) \rightarrow (\frac{k[G] \otimes A^{(n+1)}}{1-T}, b, B)$, coming from the map sending (a_0, a_1, \dots, a_n) to $(g^{-1}/a_0, a_1, \dots, a_n)$, using the fact that $\frac{A^{(n+1)}}{1-T_g} = \frac{A^{(n+1)}}{1-T_{g^{-1}}}$. This induces a map $HC_*^g(A) \rightarrow HC_*^G(A)$. From the inclusion of bicomplexes

$$(\frac{k[G] \otimes A^{(n+1)}}{1-T}, b, B) \hookrightarrow (\sum_{p+q=n} \frac{k[G^{p+1}] \otimes A^{(q+1)}}{1-T}, b + \bar{b}, B)$$

we get a map $HC_*^G(A) \rightarrow HC_*(A \rtimes G)$. Composing these two we get a map from $HC_*^g(A)$ to $HC_*(A \rtimes G)$, for any $g \in G$. If we take G to be a finite group with $|G|$ invertible in k then by [1, Proposition 4.6], $HC_*(A \rtimes G)$ is isomorphic to the homology of the bicomplex $(H_0(G, k[G] \otimes A^{(n+1)}), b, B) = (\frac{k[G] \otimes A^{(n+1)}}{G}, b, B)$. The action of G on $k[G] \otimes A^{(n+1)}$ involves conjugation on the G -factor, while $1 - T$ acts as the identity on the G -factor. In that case can think of the map $HC_*^g(A) \rightarrow HC_*(A \rtimes G)$ as being induced by

$$\frac{A^{(n+1)}}{1-T_g} \rightarrow \frac{k[G] \otimes A^{(n+1)}}{1-T} \rightarrow \frac{k[G] \otimes A^{(n+1)}}{G}.$$

Decomposing G into its conjugacy classes, $[g] = \{h^{-1}gh/h \in G\}$, and then applying Shapiro's lemma on H_0 , we get an isomorphism, $\frac{k[G] \otimes A^{(n+1)}}{G} \simeq \sum_{[g]} \frac{A^{(n+1)}}{G^g}$, where G^g is the centralizer of g in G and each $A^{(n+1)}$ over G^g represents the stalk of $k[G] \otimes A^{(n+1)}$ over g . Since for each $[g]$ the boundary maps b and B send the stalk to itself we have a quasi-isomorphism of bicomplexes, $(\frac{k[G] \otimes A^{(n+1)}}{G}, b, B) \simeq (\sum_{[g]} (\frac{A^{(n+1)}}{G^g}, b, B))$. Then for each $g \in G$ we get a composition of bicomplexes

$$(\frac{A^{(n+1)}}{1-T_g}, b, B) \rightarrow (\frac{A^{(n+1)}}{G^g}, b, B) \hookrightarrow (\sum_{[g]} \frac{A^{(n+1)}}{G^g}, b, B) \simeq (\frac{k[G] \otimes A^{(n+1)}}{G}, b, B),$$

which induces the homomorphism $HC_*^g(A) \rightarrow HC_*(A \rtimes G)$. If G is the cyclic group generated by g , then for all $g' \in G$ we have $[g'] = g'$, $G^{g'} = G$ and $\frac{A^{(n+1)}}{G} = \frac{A^{(n+1)}}{g}$, which means that $HC_*(A \rtimes G)$ is isomorphic to a direct sum of $HC_*^g(A)$, r -times, where $r = |G|$, and under the homomorphism, $HC_*^g(A)$ maps isomorphically onto one of the direct summands. \square

Corollary 1. *If g and g' are k -algebra automorphisms of A which generate the same finite cyclic group G , whose order is invertible in k , then $HC_*^g(A) \simeq HC_*^{g'}(A)$. In particular if $ch(k) = 0$ and $|g| = r$ then we have $HC_*^g(A) \simeq HC_*^{g^n}(A)$, for g raised to any power n , $(n, r) = 1$.*

For the case where G is a finite group with $|G|$ invertible in k we have the cyclic homology of $A \rtimes G$ as the homology of the bicomplex $(\frac{k[G] \otimes A^{(n+1)}}{G}, b, B)$. Using the first column of that bicomplex we can extend the definition of twisted Hochschild homology given in [2] to a full group G .

Definition 1. *The G -Hochschild homology of A , $HH_*^G(A)$, is the homology of the chain complex $(\frac{k[G] \otimes A^{(n+1)}}{G}, b)$, the first column of the bicomplex $(\frac{k[G] \otimes A^{(n+1)}}{G}, b, B)$.*

From the procedure used in [3, remark 2.2.2] we get from this a Connes' periodicity exact sequence,

$$\dots \rightarrow HH_n^G(A) \rightarrow HC_n(A \rtimes G) \rightarrow HC_{n-2}(A \rtimes G) \rightarrow HH_{n-2}^G(A) \rightarrow \dots$$

Next we can generalize the procedure used in [4] for twisted de Rham homology to define G -de Rham homology over a group, G . First let $\bar{C}_n^G(A) \equiv \frac{k[G] \otimes A \otimes \bar{A}^{(n)}}{G}$, and put on it the differential defined by $d(g/a_0, \bar{a}_1, \dots, \bar{a}_n) = (g/1, \bar{a}_0, \bar{a}_1, \dots, \bar{a}_n)$. Then let $\bar{C}_n^G(A)_{ab}$ be $\frac{\bar{C}_n^G(A)}{Im(bd+db)+Im(b)}$, which is well defined since both b and d commute with the action of G .

Definition 2. The G -de Rham homology, $\bar{H}DR_n^G(A)$, is the homology of the chain complex $(\bar{C}_*^G(A)_{ab}, d)$.

For the case where $Q \subset k$ we automatically get that $|G|$ is invertible in k . More than that we get that for a finite group G the cyclic homology $HC_*(A \rtimes G)$ is isomorphic to the homology coming from the Connes complex $(\frac{k[G] \otimes A^{(n+1)}}{1-t}, b)$, where $t(g/a_0, \dots, a_n) = (g/g^{-1}(a_n), a_0, \dots, a_{n-1})$. Using that we can generalize the Karoubi theorem proved in [4] for twisted cyclic homology. The arguments will be the same with $\bar{C}_n^G(A)$ replacing $\bar{C}_n^g(A)$.

Corollary 2. If G is a finite group, A a k -algebra with $Q \subset k$, then the G -de Rham homology of A and the cyclic homology of $A \rtimes G$ are related by the following exact sequence,

$$0 \rightarrow \bar{H}DR_n^G(A) \rightarrow \bar{H}C_n(A \rtimes G) \rightarrow \bar{H}H_{n+1}^G(A).$$

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MATH DEPARTMENT, WASHINGTON UNIVERSITY, SAINT LOUIS, MO 63130 USA

E-mail address: jshapiro@math.wustl.edu